



4.4 Norms in other vector spaces

Douglas Wilhelm Harder, LEL, M.Math.

dwharder@uwaterloo.ca

dwharder@gmail.com





Introduction

- In this topic, we will
 - Define the 2-norm for discrete and continuous signals
 - Look at some examples





2-norm of discrete signals

- Given the discrete signal

$$x = (x[0], x[1], x[2], x[3], \dots)$$

we could define $\|x\|_2 = \sqrt{\sum_{k=0}^{\infty} |x[k]|^2}$

- Unfortunately, under this definition, some discrete signals have a norm equal to infinity
 - The subset of all discrete signals that have finite norms, that is,

$$\|x\|_2 = \sqrt{\sum_{k=0}^{\infty} |x[k]|^2} < \infty$$

forms a subspace of the vector space of all discrete signals





2-norm of discrete signals

- For example,
 - If $|\gamma| < 1$, then $\gamma^n = (1, \gamma, \gamma^2, \gamma^3, \gamma^4, \dots)$ is such a signal,

$$\begin{aligned} \text{for } \|\gamma^n\|_2 &= \sqrt{\sum_{k=0}^{\infty} |\gamma^k|^2} \\ &= \sqrt{\sum_{k=0}^{\infty} |\gamma^2|^k} = \sqrt{\frac{1}{1-|\gamma|^2}} \end{aligned}$$





2-norm of discrete signals

- For example, the discrete signal

$$\frac{1}{3}^n = \left(1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots\right)$$

has a 2-norm equal to

$$\left\| \frac{1}{3}^n \right\|_2 = \sqrt{\frac{1}{1 - \frac{1}{9}}} = \sqrt{\frac{9}{8}} = \frac{3}{2\sqrt{2}}$$

- Thus, a unit discrete signal would be

$$2\sqrt{2} \left(\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \dots \right)$$





Infinity-norm of discrete signals

- Given the discrete signal

$$x = (x[0], x[1], x[2], x[3], \dots)$$

we could define $\|x\|_{\infty} = \sup_{k=0,1,2,\dots} \{|x[k]|\}$

- Here “sup” means *supremum*, which is similar to a maximum
- The subspace of all discrete signals for which this norm is finite equals the subspace of all bounded signals





2-norms of continuous signals

- Given the continuous signal f defined on $[0, \infty)$, we can define

$$\|f\|_2 = \sqrt{\int_0^{\infty} |f(t)|^2 dt}$$

- Again, a lot of functions have an infinite square integral
 - If we restrict ourselves to those functions for which

$$\|f\|_2 = \sqrt{\int_0^{\infty} |f(t)|^2 dt} < \infty$$

this forms a subspace of all continuous signals

- This is the subspace of all *square-integrable functions*





2-norms of continuous signals

- For example, if

$$f(x) \stackrel{\text{def}}{=} e^{-5t} \sin(10t)$$

so

$$\|f\|_2 = \sqrt{\int_0^{\infty} \left(e^{-5t} \sin(10t) \right)^2 dt} = \frac{1}{5}$$

- Thus, $5e^{-5t} \sin(10t)$ would now be a *unit function*





Summary

- Following this topic, you now
 - Are aware that there are norms on discrete and continuous signals
 - Understand these are analogous to the finite-dimensional norms





References

- [1] [https://en.wikipedia.org/wiki/Norm_\(mathematics\)](https://en.wikipedia.org/wiki/Norm_(mathematics))





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.





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